

# An Averaging Theorem for Perturbed KdV Equation

HUANG Guan

C.M.L.S, Ecole Polytechnique, Palaiseau, France

E-mail: [guan@math.polytechnique.fr](mailto:guan@math.polytechnique.fr)

**Abstract.** We consider a perturbed KdV equation:

$$\dot{u} + u_{xxx} - 6uu_x = \epsilon f(x, u(\cdot)), \quad x \in \mathbb{T}, \quad \int_{\mathbb{T}} u dx = 0.$$

For any periodic function  $u(x)$ , let  $I(u) = (I_1(u), I_2(u), \dots) \in \mathbb{R}_+^\infty$  be the vector, formed by the KdV integrals of motion, calculated for the potential  $u(x)$ . Assuming that the perturbation  $\epsilon f(x, u(\cdot))$  is a smoothing mapping (e.g. it is a smooth function  $\epsilon f(x)$ , independent from  $u$ ), and that solutions of the perturbed equation satisfy some mild a-priori assumptions, we prove that for solutions  $u(t, x)$  with typical initial data and for  $0 \leq t \lesssim \epsilon^{-1}$ , the vector  $I(u(t))$  may be well approximated by a solution of the averaged equation.

AMS classification scheme numbers: 35Q53, 70K65, 34C29, 37K10, 74H40

## 0. Introduction

We consider a perturbed Korteweg-de Vries (KdV) equation with zero mean-value periodic boundary condition:

$$\dot{u} + u_{xxx} - 6uu_x = \epsilon f(x, u(\cdot)), \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, \quad \int_{\mathbb{T}} u(x, t) dx = 0. \quad (0.1)$$

Here  $\epsilon f(x, u(\cdot))$  is a nonlinear perturbation, specified below. For any  $p \in \mathbb{R}$  we denote by  $H^p$  the Sobolev space of order  $p$ , formed by real-valued periodic functions with zero mean-value, provided with the homogeneous norm  $\|\cdot\|_p$ . Particularly, if  $p \in \mathbb{N}$  we have

$$H^p = \left\{ u \in L^2(\mathbb{T}) : \|u\|_p < \infty, \int_{\mathbb{T}} u dx = 0 \right\}, \quad \|u\|_p^2 = \int_{\mathbb{T}} \left| \frac{\partial^p u}{\partial x^p} \right|^2 dx.$$

For any  $p$ , the operator  $\frac{\partial}{\partial x}$  defines a linear isomorphism:  $\frac{\partial}{\partial x} : H^p \rightarrow H^{p-1}$ . Denoting by  $(\frac{\partial}{\partial x})^{-1}$  its inverse, we provide the spaces  $H^p$ ,  $p \geq 0$ , with a symplectic structure by means of the 2-form  $\Omega$ :

$$\Omega(u_1, u_2) = -\left\langle \left(\frac{\partial}{\partial x}\right)^{-1} u_1, u_2 \right\rangle, \quad (0.2)$$

where  $\langle \cdot, \cdot \rangle$  is the scalar product in  $L^2(\mathbb{T})$ . Then in any space  $H^p$ ,  $p \geq 1$ , the KdV equation  $(0.1)_{\epsilon=0}$  may be written as a Hamiltonian system with the Hamiltonian  $\mathcal{H}$ , given by  $\mathcal{H}(u) = \int_{\mathbb{T}} \left( \frac{1}{2} u_x^2 + u^3 \right) dx$ . That is, KdV may be written as

$$\dot{u} = \frac{\partial}{\partial x} \nabla \mathcal{H}(u).$$

It is well-known that KdV is integrable. It means that the function space  $H^p$  admits analytic symplectic coordinates  $v = (\mathbf{v}_1, \mathbf{v}_2, \dots) = \Psi(u(\cdot))$ , where  $\mathbf{v}_j = (v_j, v_{-j}) \in \mathbb{R}^2$ , such that the quantities  $I_j = \frac{1}{2} |\mathbf{v}_j|^2$ ,  $j \geq 1$ , are actions (integrals of motion), while  $\varphi_j = \text{Arg } \mathbf{v}_j$ ,  $j \geq 1$ , are angles. In the  $(I, \varphi)$ -variables, KdV takes the integrable form

$$\dot{I} = 0, \quad \dot{\varphi} = W(I), \quad (0.3)$$

where  $W(I) \in \mathbb{R}^\infty$  is the frequency vector (see [1, 2]). The integrating transformation  $\Psi$ , called the nonlinear Fourier transform, for any  $p \geq 0$  defines an analytic isomorphism  $\Psi : H^p \rightarrow h^p$ , where

$$h^p = \left\{ v = (\mathbf{v}_1, \mathbf{v}_2, \dots) : |v|_p^2 = \sum_{j=1}^{+\infty} (2\pi j)^{2p+1} |\mathbf{v}_j|^2 < \infty, \mathbf{v}_j \in \mathbb{R}^2, j \in \mathbb{N} \right\}.$$

It is well established that for a perturbed integrable finite-dimensional system,

$$\dot{I} = \epsilon f(I, \varphi), \quad \dot{\varphi} = W(I) + \epsilon g(I, \varphi), \quad \epsilon \ll 1,$$

where  $I \in \mathbb{R}^n$ ,  $\varphi \in \mathbb{T}^n$ , on time intervals of order  $\epsilon^{-1}$  the actions  $I(t)$  may be well approximated by solutions of the averaged equation:

$$\dot{J} = \epsilon \langle f \rangle(J), \quad \langle f \rangle(J) = \int_{\mathbb{T}^n} f(J, \varphi) d\varphi,$$

provided that the initial data  $(I(0), \varphi(0))$  are typical (see [3, 4, 5, 6]). This assertion is known as the *averaging principle*. But in the infinite dimensional case, there is no

similar general result. In [7, 8], S. Kuksin and A. Piatniski proved that the averaging principle holds for the randomly perturbed KdV equation of the form:

$$\dot{u} - \epsilon u_{xx} + u_{xxx} - 6uu_x = \sqrt{\epsilon}\eta(t, x), \quad x \in \mathbb{S}^1, \quad \int u dx = \int \eta dx = 0, \quad (0.4)$$

where the force  $\eta$  is a white noise in  $t$ , is smooth in  $x$  and is non-degenerate. Our goal in this work is to justify the averaging principle for the KdV equation with deterministic perturbations, using the Anosov scheme (see [3]), exploited earlier in the finite dimensional situation. The main technical difficulty to achieve this goal comes from the fact that to perform the scheme one has to use a measure in the function space which is quasi-invariant under the flow of the perturbed equation (it is needed to guarantee that a small 'bad' set which we have to prohibit for a solution of the perturbed equation at a time  $t > 0$  corresponds to a small set of initial data). For a reason, explained in Section 3, to construct such a quasi-invariant measure we have to assume that the perturbation  $\epsilon f$  is smoothing. More precisely, we assume that:

**Assumption A.** (i) For any  $p \geq 0$ , the mapping defined by the perturbation in (0.1):

$$\mathcal{P} : H^p \rightarrow H^{p+\zeta_0}, \quad u \mapsto f(x, u(\cdot)), \quad (0.5)$$

is analytic. Here  $\zeta_0 > 1$  is a constant.

(ii) For any  $p \geq 3$  and  $T > 0$ , the perturbed KdV equation (0.1) with initial data

$$u(0) = u_0 \in H^p,$$

has a unique solution  $u(t, x) \in H^p$  in the time interval  $[-T\epsilon^{-1}, T\epsilon^{-1}]$ , and

$$\|u(t)\|_p \leq C(p, \|u_0\|_p, T), \quad |t| \leq T\epsilon^{-1}.$$

We are mainly concerned with the behavior of the actions  $I(u(t)) \in \mathbb{R}_+^\infty$  for  $|t| \lesssim \epsilon^{-1}$ . For this end, it is convenient to pass to the slow time  $\tau = \epsilon t$  and write the perturbed KdV equation (0.1) in the action-angle coordinates  $(I, \varphi)$ :

$$\frac{dI}{d\tau} = F(I, \varphi), \quad \frac{d\varphi}{d\tau} = \epsilon^{-1}W(I) + G(I, \varphi). \quad (0.6)$$

Here  $I \in \mathbb{R}^\infty$ ,  $\varphi \in \mathbb{T}^\infty$  and  $\mathbb{T}^\infty := \{\theta = (\theta_i)_{i \geq 1}, \theta_i \in \mathbb{T}\}$  is the infinite-dimensional torus, endowed with the Tikhonov topology. The two functions  $F(I, \varphi)$  and  $G(I, \varphi)$  are the perturbation term  $\epsilon f$ , written in action-angle variables, see below (1.3) and (1.4). The corresponding averaged equation is

$$\frac{dJ}{d\tau} = \langle F \rangle(J), \quad \langle F \rangle(J) = \int_{\mathbb{T}^\infty} F(J, \varphi) d\varphi, \quad (0.7)$$

where  $d\varphi$  is the Haar measure on  $\mathbb{T}^\infty$ . It turns out that the (0.7) is a Lipschitz equation, see below (4.17). We denote by  $h_{I+}^p$  the image of the space  $h^p$  under the action-mapping

$$\pi_I : v \mapsto I, \quad I_j(v) = \frac{1}{2} |\mathbf{v}_j|^2, \quad j \geq 1.$$

Clearly,  $I = \pi_I(v) \in h_{I+}^p \subset h_I^p$ , where  $h_I^p$  is the weighted  $l^1$ -space

$$h_I^p = \left\{ I \in \mathbb{R}^\infty : |I|_{h_I^p} = |I|_p = 2 \sum_{j=1}^{\infty} (2\pi j)^{2p+1} |I_j| < \infty \right\}.$$

and  $h_{I+}^p$  is its positive octant,  $h_{I+}^p = \{I \in h_I^p : I_j \geq 0, \forall j\}$ . This is a closed subset of  $h_I^p$ .

For any  $\theta = (\theta_i)_{i \geq 1} \in \mathbb{T}^\infty$ , let us denote by  $\Phi_\theta$  the linear operator on the space of sequences  $(\mathbf{v}_1, \mathbf{v}_2, \dots) \in h^p$  which rotates each component  $\mathbf{v}_j \in \mathbb{R}^2$  by the angle  $\theta_j$ .

**Definition 0.1** A Gaussian measure  $\mu$  on the Hilbert space  $h^p$  is said to be  $\zeta_0$ -admissible (where  $\zeta_0 > 1$  is the same as in assumption A), if the following conditions are fulfilled:

- (i) It is non-degenerate and has zero mean value.
- (ii) It has a diagonal correlation operator  $(\mathbf{v}_1, \mathbf{v}_2, \dots) \mapsto (\sigma_1 \mathbf{v}_1, \sigma_2 \mathbf{v}_2, \dots)$ , where every  $\sigma_j > 0$ ,  $\sum_{j \geq 1} \sigma_j < \infty$  and  $j^{-\zeta_0}/\sigma_j = O(1)$ . In particular,  $\mu$  is invariant under the rotations  $\Phi_\theta$ .

Such measures can be written as:

$$\prod_{j=1}^{+\infty} \frac{(2\pi j)^{1+2p}}{2\pi\sigma_j} \exp\left\{-\frac{(2\pi j)^{1+2p}|\mathbf{v}_j|^2}{2\sigma_j}\right\} d\mathbf{v}_j, \quad (0.8)$$

where  $d\mathbf{v}_j$ ,  $j \geq 1$ , is the Lebesgue measure on  $\mathbb{R}^2$  (see [9, 10]). Clearly, they are invariant under the KdV flow (0.3).

The main result of this work is the following theorem:

**Theorem 0.2.** Fix any  $p \geq 3$  and  $\bar{T} > 0$ . Let the curve  $u^\epsilon(t) \in H^p$ ,  $|t| \leq \epsilon^{-1}\bar{T}$  be a solution of equation (0.1) and  $v^\epsilon(\tau) = \Psi(u^\epsilon(\epsilon^{-1}\tau))$ ,  $\tau = \epsilon t$ ,  $|\tau| \leq \bar{T}$ . If assumption A is fulfilled and  $\mu$  is a  $\zeta_0$ -admissible Gaussian measure on  $h^p$ , then

- (i) For any  $\rho > 0$ , there exists a Borel subset  $\Gamma_\rho^\epsilon$  of  $h^p$  and  $\epsilon_\rho > 0$  such that  $\lim_{\epsilon \rightarrow 0} \mu(h^p \setminus \Gamma_\rho^\epsilon) = 0$ , and for  $\epsilon \leq \epsilon_\rho$  we have

$$|I(v^\epsilon(\tau)) - J(\tau)|_p \leq \rho, \quad \text{for } |\tau| \leq \bar{T}, \quad v^\epsilon(0) \in \Gamma_\rho^\epsilon, \quad (0.9)$$

where  $J(\tau)$ ,  $|\tau| \leq \bar{T}$ , is a solution of the averaged equation (0.7) with the initial data  $J(0) = \pi_I(v^\epsilon(0))$ .

- (ii) There is a full measure subset  $\Gamma_\varphi$  of  $h^p$  with the following property: If  $v^\epsilon(0) \in \Gamma_\varphi$ , then for any  $0 \leq \bar{T}_1 < \bar{T}_2 \leq \bar{T}$  the image  $\mu_{\bar{T}_1, \bar{T}_2}^\epsilon$  of the probability measure  $(\bar{T}_2 - \bar{T}_1)^{-1} d\tau$  on  $[\bar{T}_1, \bar{T}_2]$  under the mapping  $\tau \mapsto \varphi(v^\epsilon(\tau)) \in \mathbb{T}^\infty$  converges weakly, as  $\epsilon \rightarrow 0$ , to the Haar measure  $d\varphi$  on  $\mathbb{T}^\infty$ .

The assertion (ii) of the theorem means that for any bounded continuous function  $g(\varphi)$  on  $\mathbb{T}^\infty$ ,

$$\frac{1}{\bar{T}_2 - \bar{T}_1} \int_{\bar{T}_1}^{\bar{T}_2} g(\varphi(v^\epsilon(\tau))) d\tau \rightarrow \int_{\mathbb{T}^\infty} g(\varphi) d\varphi, \quad \epsilon \rightarrow 0.$$

In particular, we have

**Proposition 0.3.** The assumption A holds if in (0.1)  $f = f(x)$  is a smooth function, independent from  $u$ .

It is unknown for us that if the result of Theorem 0.2 remains true for equation (0.1) with non-smoothing perturbations, e.g. if the right hand side of equation (0.1) is  $\epsilon u_{xx}$  or  $-\epsilon u$ . So we do not know whether a suitable analogy of the result in [7, 8] holds true if in equation (0.4) the noise  $\eta$  vanishes.

The paper has the following structure: Section 1 is about the transformation which integrates the KdV and its Birkhoff normal form. In Section 2 we discuss the averaged equation. We prove that the  $\zeta_0$ -admissible Gaussian measures are quasi-invariant under the flow of equation (0.1) in Section 3. Finally in Section 4 and Section 5 we establish the main theorem and Proposition 0.3.

**Agreements.** Analyticity of maps  $B_1 \rightarrow B_2$  between Banach spaces  $B_1$  and  $B_2$ , which are the real parts of complex spaces  $B_1^c$  and  $B_2^c$ , is understood in the sense of Fréchet. All analytic maps that we consider possess the following additional property: for any  $R$ , a map extends to a bounded analytical mapping in a complex  $(\delta_R > 0)$ -neighborhood of the ball  $\{|u|_{B_1} < R\}$  in  $B_1^c$ .

**Notation.** We use capital letters  $C$  or  $C(a_1, a_2, \dots)$  to denote positive constants that depend on the parameters  $a_1, a_2, \dots$  but not on the unknown function  $u$ . We denote  $\mathbb{Z}_{\geq 0} = \{n \in \mathbb{Z}, n \geq 0\}$ . For an infinite-dimensional vector  $w = (w_1, w_2, \dots)$  and any  $n \in \mathbb{N}$  we denote  $w^n = (w_1, \dots, w_n, 0, 0, \dots)$ . We often identify  $w^n$  with a corresponding  $n$ -vector.

## 1. Preliminaries on the KdV equation

In this section we discuss integrability of the KdV equation  $(0.1)_{\epsilon=0}$ .

### 1.1. Nonlinear Fourier transform for KdV

We provide the  $L^2$ -space  $H^0$  with the Hilbert basis  $\{e_s, s \in \mathbb{Z} \setminus \{0\}\}$ ,

$$e_s = \begin{cases} \sqrt{2} \cos(2\pi s x) & s > 0, \\ \sqrt{2} \sin(2\pi s x) & s < 0. \end{cases}$$

**Theorem 1.1.** There exists an analytic diffeomorphism  $\Psi : H^0 \mapsto h^0$  and an analytic functional  $K$  on  $h^0$  of the form  $K(v) = \tilde{K}(I(v))$ , where the function  $\tilde{K}(I)$  is analytic in a suitable neighborhood of the octant  $h_{I+}^0$  in  $h_I^0$ , with the following properties:

(i) The mapping  $\Psi$  defines an analytic diffeomorphism  $\Psi : H^p \mapsto h^p$ , for any  $p \in \mathbb{Z}_{\geq 0}$ . This is a symplectomorphism of the spaces  $(H^p, \Omega)$  (see (0.2) and  $(h^p, \omega_2)$ , where  $\omega_2 = \sum dv_k \wedge dv_{-k}$ .

(ii) The differential  $d\Psi(0)$  takes the form  $\sum u_s e_s \mapsto v, v_s = |2\pi s|^{-1/2} u_s$ .

(iii) A curve  $u \in C^1(0, T; H^0)$  is a solution of the KdV equation  $(0.1)_{\epsilon=0}$  if and only if  $v(t) = \Psi(u(t))$  satisfies the equation

$$\dot{\mathbf{v}}_j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial \tilde{K}}{\partial I_j}(I) \mathbf{v}_j, \quad \mathbf{v}_j = (v_j, v_{-j}) \in \mathbb{R}^2, \quad j \in \mathbb{N}. \quad (1.1)$$

Since the maps  $\Psi$  and  $\Psi^{-1}$  are analytic, then for  $m = 0, 1, 2, \dots$ , we have

$$\|d^j \Psi(u)\|_m \leq P_m(\|u\|_m), \quad \|d^j \Psi^{-1}(v)\|_m \leq Q_m(\|v\|_m), \quad j = 0, 1, 2,$$

where  $P_m$  and  $Q_m$  are continuous functions (cf. the agreements).

We denote

$$W(I) = (W_1, W_2, \dots), \quad W_k(I) = \frac{\partial \tilde{K}}{\partial I_k}(I), \quad k = 1, 2, \dots$$

**Lemma 1.2.** For any  $n \in \mathbb{N}$ , if  $I_{n+1} = I_{n+2} = \dots = 0$ , then

$$\det\left(\left(\frac{\partial W_i}{\partial I_j}\right)_{1 \leq i, j \leq n}\right) \neq 0.$$

Let  $l_{-1}^\infty$  be the Banach space of all real sequences  $l = (l_1, l_2, \dots)$  with the norm

$$|l|_{-1} = \sup_{n \geq 1} n^{-1} |l_n| < \infty.$$

Denote  $\kappa = (\kappa_n)_{n \geq 1}$ , where  $\kappa_n = (2\pi n)^3$ .

**Lemma 1.3.** The normalized frequency map

$$\tilde{W} : I \mapsto \tilde{W}(I) = W(I) - \kappa$$

is real analytic as a map from  $h_{I+}^1$  to  $l_{-1}^\infty$ .

The coordinates  $v = \Psi(u)$  are called the *Birkhoff coordinates*, and the form (1.1) of KdV is its *Birkhoff normal form*. See [1] for Theorem 1.1 and Lemma 1.3. A detailed proof of Lemma 1.2 can be found in [2].

### 1.2. Equation (0.1) in the Birkhoff coordinates.

For  $k = 1, 2, \dots$  we denote:

$$\Psi_k : H^m \rightarrow \mathbb{R}^2, \quad \Psi_k(u) = \mathbf{v}_k,$$

where  $\Psi(u) = v = (\mathbf{v}_1, \mathbf{v}_2, \dots)$ . Let  $u(t)$  be a solution of equation (0.1). We get

$$\dot{\mathbf{v}}_k = d\Psi_k(u)(\epsilon f(x, u) + V(u)), \quad k \geq 1, \quad (1.2)$$

where  $V(u) = -u_{xxx} + 6uu_x$ . Since  $I_k(v) = \frac{1}{2}|\Psi_k|^2$  is an integral of motion of KdV equation (0.1) $_{\epsilon=0}$ , we have

$$\dot{I}_k = \epsilon(d\Psi_k(u)f(x, u), \mathbf{v}_k) := \epsilon F_k(v). \quad (1.3)$$

Here and below  $(\cdot, \cdot)$  indicates the scalar product in  $\mathbb{R}^2$ .

For  $k \geq 1$  define  $\varphi_k = \arctan(\frac{v_{-k}}{v_k})$  if  $\mathbf{v}_k \neq 0$ , and  $\varphi_k = 0$  if  $\mathbf{v}_k = 0$ . Using equation (1.1), we get

$$\dot{\varphi}_k = W_k(I) + \epsilon|\mathbf{v}_k|^{-2}(d\Psi_k(u)f(x, u), \mathbf{v}_k^\perp), \quad \text{if } \mathbf{v}_k \neq 0, \quad (1.4)$$

where  $\mathbf{v}_k^\perp = (-v_{-k}, v_k)$ . Denoting for brevity, the vector field in equation (1.4) by  $W_k(I) + \epsilon G_k(v)$ , we rewrite the equation for the pair  $(I_k, \varphi_k)(k \geq 1)$  as

$$\begin{aligned} \dot{I}_k(t) &= \epsilon F_k(v) = \epsilon F_k(I, \varphi), \\ \dot{\varphi}_k(t) &= W_k(I) + \epsilon G_k(v). \end{aligned} \quad (1.5)$$

We set

$$F(I, \varphi) = (F_1(I, \varphi), F_2(I, \varphi), \dots).$$

In the following lemma  $P_k$  and  $P_k^j$  are some fixed continuous functions.

**Lemma 1.4.** For  $k, j \in \mathbb{N}$ , we have for any  $p \geq 0$

- (i) The function  $F_k(v)$  is analytic in each space  $h^p$ .
- (ii) For any  $p \geq 0$ ,  $\delta > 0$ , the function  $G_k(v)\chi_{\{I_k \geq \delta\}}$  is bounded by  $\delta^{-1/2}P_k(|v|_p)$ .
- (iii) For any  $\delta > 0$ , the function  $\frac{\partial F_k}{\partial I_j}(I, \varphi)\chi_{\{I_j \geq \delta\}}$  is bounded by  $\delta^{-1/2}P_k^j(|v|_p)$ .
- (iv) The function  $\frac{\partial F_k}{\partial \varphi_j}(I, \varphi)$  is bounded by  $P_k^j(|v|_p)$ , and for any  $n \in \mathbb{N}$  and  $(I_1, \dots, I_n) \in \mathbb{R}_+^n$ , the function  $F_k(I_1, \varphi_1, \dots, I_n, \varphi_n, 0, \dots)$  is analytic on  $\mathbb{T}^n$ .

*Proof:* Items (i) and (ii) follow directly from Theorem 1.1. Items (iii) and (iv) follow from item (i) and the chain-rule:

$$\begin{aligned}\frac{\partial F_k}{\partial \varphi_j} &= \sqrt{2I_j} \left( \frac{\partial F_k}{\partial v_{-j}} \cos(\varphi_j) - \frac{\partial F_k}{\partial v_j} \sin(\varphi_j) \right), \\ \frac{\partial F_k}{\partial I_j} &= (\sqrt{2I_j})^{-1} \left( \frac{\partial F_k}{\partial v_j} \cos(\varphi_j) + \frac{\partial F_k}{\partial v_{-j}} \sin(\varphi_j) \right). \quad \square\end{aligned}$$

From this lemma we know that equation (1.5) may have singularities at  $\partial h_{I+}^p$ . We denote

$$\begin{aligned}\Pi_I : h^p &\rightarrow h_I^p, \quad \Pi_I(v) = I(v), \\ \Pi_{I,\varphi} : h^p &\rightarrow h_I^p \times \mathbb{T}^\infty, \quad \Pi_{I,\varphi}(v) = (I(v), \varphi(v)).\end{aligned}$$

Abusing notation, we will identify  $v$  with  $(I, \varphi) = \Pi_{I,\varphi}(v)$ .

**Definition 1.5.** For  $p \geq 3$ , we say that a curve  $(I(t), \varphi(t))$ ,  $|t| \leq T$ , is a regular solution of equation (1.5), if there exists a solution  $u(t) \in H^p$  of equation (0.1) such that  $u(t) \in H^p$  and

$$\Pi_{I,\varphi}(\Psi(u(t))) = (I(t), \varphi(t)), \quad |t| \leq T.$$

If  $(I(t), \varphi(t))$  is a regular solution of (1.5) and  $|I(0)|_p \leq M_0$ , then by assumption A we have

$$|I(t)|_p = |v(t)|_p^2 \leq C(p, M_0, T), \quad |t| \leq T\epsilon^{-1}. \quad (1.6)$$

## 2. Averaged equation

For a function  $f$  on a Hilbert space  $H$ , we write  $f \in Lip_{loc}(H)$  if

$$|f(u_1) - f(u_2)| \leq P(R) \|u_1 - u_2\|, \quad \|u_1\|, \|u_2\| \leq R, \quad (2.1)$$

for a suitable continuous function  $P$  which depends on  $f$ . Clearly, the set of functions  $Lip_{loc}(H)$  is an algebra. By the Cauchy inequality, any analytic function on  $H$  belongs to  $Lip_{loc}(H)$  (see agreements). In particular, for any  $k \geq 1$ ,

$$W_k(I) \in Lip_{loc}(h_I^p), \quad p \geq 1, \quad \text{and} \quad F_k(v) \in Lip_{loc}(h^p), \quad p \geq 0.$$

In the further analysis, we systematically use the fact that the functional  $F_k(v)$  only weakly depends on the tail of the vector  $v$ . Now we state the corresponding results. Let  $f \in Lip_{loc}(h^p)$  and  $v \in h^{p_1}$ ,  $p_1 > p$ . Denoting by  $\Pi^M$ ,  $M \geq 1$  the projection

$$\Pi^M : h^0 \rightarrow h^0, \quad (\mathbf{v}_1, \mathbf{v}_2, \dots) \mapsto (\mathbf{v}_1, \dots, \mathbf{v}_M, 0, \dots),$$

we have  $|v - \Pi^M v|_p \leq (2\pi M)^{-(p_1-p)} |v|_{p_1}$ . Accordingly,

$$|f(v) - f(\Pi^M v)| \leq P(|v|_{p_1})(2\pi M)^{-(p_1-p)}. \quad (2.2)$$

The torus  $\mathbb{T}^M$  acts on the space  $\Pi_M h^0$  by linear transformations  $\Phi_{\theta_M}$ ,  $\theta_M \in \mathbb{T}^M$ , where  $\Phi_{\theta_M} : (I_M, \varphi_M) \mapsto (I_M, \varphi_M + \theta_M)$ . Similarly, the torus  $\mathbb{T}^\infty$  acts on  $h^0$  by linear transformations  $\Phi_\theta : (I, \varphi) \mapsto (I, \varphi + \theta)$  with  $\theta \in \mathbb{T}^\infty$ .

For a function  $f \in Lip_{loc}(h^p)$  and a positive integer  $N$  we define the average of  $f$  in the first  $N$  angles as the function

$$\langle f \rangle_N(v) = \int_{\mathbb{T}^N} f((\Phi_{\theta_N} \oplus \text{Id})(v)) d\theta_N,$$

and define the averaging in all angles as

$$\langle f \rangle(v) = \int_{\mathbb{T}^\infty} f(\Phi_\theta(v)) d\theta,$$

where  $d\theta$  is the Haar measure on  $\mathbb{T}^\infty$ . The estimate (2.2) readily implies that

$$|\langle f \rangle_N(v) - \langle f \rangle(v)| \leq P(R)(2\pi N)^{-(p_1-p)}, \quad |v|_{p_1} \leq R.$$

Let  $v = (I, \varphi)$ , then  $\langle f \rangle_N$  is a function independent of  $\varphi_1, \dots, \varphi_N$ , and  $\langle f \rangle$  is independent of  $\varphi$ . Thus  $\langle f \rangle$  can be written as  $\langle f \rangle(I)$ .

**Lemma 2.1.** (See [7]). Let  $f \in Lip_{loc}(h^p)$ , then

- (i) The functions  $\langle f \rangle_N(v)$  and  $\langle f \rangle(v)$  satisfy (2.1) with the same function  $P$  as  $f$  and take the same value at the origin.
- (ii) These two functions are smooth (analytic) if  $f$  is. If  $f$  is smooth, then  $\langle f \rangle(I)$  is a smooth function with respect to vector  $(I_1, \dots, I_M)$ , for any  $M$ . If  $f(v)$  is analytic in the space  $h^p$ , then  $\langle f \rangle(I)$  is analytic in the space  $h_I^p$ .

We recall that a vector  $\omega \in \mathbb{R}^n$  is *non-resonant* if

$$\omega \cdot k \neq 0, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}.$$

Denote by  $C^{0+1}(\mathbb{T}^n)$  the set of all Lipschitz functions on  $\mathbb{T}^n$ .

**Lemma 2.2.** Let  $f \in C^{0+1}(\mathbb{T}^n)$  for some  $n \in \mathbb{N}$ . Then for any non-resonant vector  $\omega \in \mathbb{R}^n$  we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x_0 + \omega t) dt = \langle f \rangle,$$

uniformly in  $x_0 \in \mathbb{T}^n$ . The rate of convergence depends on  $n$ ,  $\omega$  and  $f$ .

*Proof.* Let us write  $f(x)$  as the Fourier series  $f(x) = \sum f_k e^{ik \cdot x}$ . Since the Fourier series of a Lipschitz function converges uniformly (see [11]), for any  $\epsilon > 0$  we may find  $R = R_\epsilon$  such that  $\left| \sum_{|k| > R} f_k e^{ik \cdot x} \right| \leq \frac{\epsilon}{2}$  for all  $x$ . Now it is enough to show that

$$\left| \frac{1}{T} \int_0^T f_R(x_0 + \omega t) dt - f_0 \right| \leq \frac{\epsilon}{2}, \quad \forall T \geq T_\epsilon, \quad (2.3)$$



for a suitable  $T_\epsilon$ , where  $f_R(x) = \sum_{|k| \leq R} f_k e^{ik \cdot x}$ . Observing that

$$\left| \frac{1}{T} \int_0^T e^{ik \cdot (x_0 + \omega t)} dt \right| \leq \frac{2}{T|k \cdot \omega|},$$

for each nonzero  $k$ . Therefore the l.h.s of (2.3) is smaller than

$$\frac{2}{T} \left( \inf_{|k| \leq R} |k \cdot \omega| \right)^{-1} \sum_{|k| \leq R} |f_k|.$$

The assertion of the lemma follows.  $\square$

### 3. Quasi-invariance of Gaussian measures

Fix any integer  $p \geq 3$ , and let  $\mu$  be a  $\zeta_0$ -admissible Gaussian measure on the Hilbert space  $h^p$ . In this section we will discuss how this measure evolves under the flow of the perturbed KdV equation (0.1). We follow a classical procedure based on finite dimensional approximations (see e.g. [12, 10]).

We suppose the assumption A holds. Let us write the equation (0.1) in the Birkhoff normal form, using the slow time  $\tau = \epsilon t$ :

$$\frac{d}{d\tau} \mathbf{v}_j = \epsilon^{-1} \mathcal{J} W_j(I) \mathbf{v}_j + \mathbf{X}_j(v), \quad j \in \mathbb{N}, \quad (3.1)$$

where  $\mathbf{X}_j = (X_j, X_{-j})^t \in \mathbb{R}^2$  and  $\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ .

For any  $n \in \mathbb{N}$ , we consider the  $2n$ -dimensional subspace  $\pi_n(h^p)$  of  $h^p$  with coordinates  $v^n = (\mathbf{v}_1, \dots, \mathbf{v}_n, 0, \dots)$ . On  $\pi_n(h^p)$ , we define the following finite-dimensional systems:

$$\frac{d}{d\tau} \vec{\omega}_j = \epsilon^{-1} \mathcal{J} W_j(I(\omega^n)) \vec{\omega}_j + \mathbf{X}_j(\omega^n), \quad 1 \leq j \leq n, \quad (3.2)$$

where  $\vec{\omega}_j = (\omega_j, \omega_{-j})^t \in \mathbb{R}^2$  and  $\omega^n = (\vec{\omega}_1, \dots, \vec{\omega}_n, 0, \dots) \in \pi_n(h^p)$ .

We denote  $X^n(v^n) = (\mathbf{X}_1(v^n), \dots, \mathbf{X}_n(v^n), 0, \dots)$  and  $X(v) = (\mathbf{X}_1(v), \dots)$ . By assumption A and Theorem 1.1, for any  $p \geq 0$  the mapping

$$X : h^p \rightarrow h^{p+\zeta_0}, \quad v \mapsto X(v) \text{ is analytic.} \quad (3.3)$$

**Theorem 3.2.** For any  $T > 0$ ,  $\omega^n(\cdot)$  converges to  $v(\cdot)$  as  $n \rightarrow \infty$  in  $C([-T, T]; h^p)$ , where  $v(\cdot)$  and  $\omega^n(\cdot)$  are, respectively, solutions of (3.1) and (3.2) with initial data  $v(0) \in h^p$  and  $\omega^n(0) = v^n(0) \in \pi_n(h^p)$ .

*Proof.* Fix any  $M_0 > 0$ . From (1.6) we know that there exists a constant  $M_1$  such that if  $|v(0)|_p \leq M_0$ , then

$$|v(\tau)|_p \leq M_1, \quad \tau \in [0, T]. \quad (3.4)$$

The equation (3.2) yields that

$$\frac{d}{d\tau} |\omega^n|_p^2 = 2 \sum_{j=1}^n j^{1+2p} \vec{\omega}_j \cdot \mathbf{X}_j(\omega^n) := \chi^n(\omega^n). \quad (3.5)$$

We define

$$\chi(v) := 2 \sum_{j=1}^{\infty} j^{1+2p} \mathbf{v}_j \cdot \mathbf{X}_j(v).$$

By (3.3), we know that there exists a constant  $C_1 > 0$  such that

$$|\chi^n(\omega^n)| \leq C_1, \quad |\omega^n|_p \leq 2M_1, \quad \forall n \in \mathbb{N}. \quad (3.6)$$

Denote  $\bar{\tau} = M_1/C_1$ , then if  $|\omega^n(0)|_p \leq M_0$ , then

$$|\omega^n(\tau)|_p \leq 2M_1, \quad \tau \in [-\bar{\tau}, \bar{\tau}], \quad \forall n \in \mathbb{N}. \quad (3.7)$$

**Lemma 3.3.** In the space  $C([-\bar{\tau}, \bar{\tau}], h^{p-1})$ , we have the convergence

$$\omega^n(\cdot) \rightarrow v(\cdot) \quad \text{as } n \rightarrow \infty.$$

*Proof.* Denote  $\vec{\xi}_j = \mathbf{v}_j - \vec{\omega}_j$ ,  $I_v = I(v)$  and  $I_{\omega^n} = I(\omega^n)$ . Since  $\mathcal{J}\mathbf{v}_j = \mathbf{v}_j^\perp$ , using equations (3.1) and (3.2), for  $1 \leq j \leq n$ , we get

$$\begin{aligned} \frac{d}{d\tau} |\vec{\xi}_j|^2 &= 2(\vec{\xi}_j)^t [\epsilon^{-1} \mathcal{J}(W_j(I_v)\mathbf{v}_j - W_j(I_{\omega^n})\vec{\omega}_j) + \mathbf{X}_j(v) - \mathbf{X}_j(\omega^n)] \\ &= 2\epsilon^{-1} [W_j(I_v) - W_j(I_{\omega^n})] \mathbf{v}_j \cdot (\vec{\omega}_j)^\perp + 2(\vec{\xi}_j)^t \cdot (\mathbf{X}_j(v) - \mathbf{X}_j(\omega^n)). \end{aligned}$$

By Lemma 1.3 and Cauchy's inequality, we know that

$$\left| W_j(I(v)) - W_j(I(\omega^n)) \right| \leq C_2(M_1)j|v - \omega^n|_{p-1}.$$

Using (3.3) we get that

$$\frac{d}{d\tau} |v - \omega^n|_{p-1}^2 \leq C_3(\epsilon, M_1)|v - \omega^n|_{p-1}^2 + a_n(v), \quad \tau \in [-\bar{\tau}, \bar{\tau}],$$

where

$$a_n(v) = \sum_{j=n+1}^{\infty} j^{2p-1} \mathbf{v}_j \cdot \mathbf{X}_j(v).$$

Obviously,  $a_n(v) \rightarrow 0$  as  $n \rightarrow \infty$  uniformly for  $|v|_p \leq M_1$ .

The lemma now follows directly from Gronwall's Lemma.  $\square$

**Lemma 3.4.** If  $\omega^n(0) \rightarrow v(0)$  strongly in  $h^p$  and  $\tau_n \rightarrow \tau$ ,  $\tau_n \in [-\bar{\tau}, \bar{\tau}]$ , as  $n \rightarrow \infty$ , then

$$\lim_{n \rightarrow \infty} |v(\tau) - \omega^n(\tau_n)|_p = 0.$$

*Proof.* From (3.5) we know that for any  $\tau_n \in [-\bar{\tau}, \bar{\tau}]$ ,

$$|\omega^n(\tau_n)|_p^2 - |\omega^n(0)|_p^2 = \int_0^{\tau_n} \chi^n(\omega^n(s)) ds.$$

Since  $\omega^n(0) \rightarrow v(0)$  strongly in  $h^p$ , then using (3.3) and Lemma 3.3 we get

$$\begin{aligned} |v(\tau)|_p^2 &\leq \liminf_{n \rightarrow \infty} |\omega^n(\tau_n)|_p^2 \leq \limsup_{n \rightarrow \infty} |\omega^n(\tau_n)|_p^2 \\ &= \limsup_{n \rightarrow \infty} \left( |\omega^n(0)|_p^2 + \int_0^{\tau_n} \chi^n(\omega^n(s)) ds \right) = |v(0)|_p^2 + \int_0^\tau \chi(v(s)) ds \\ &= |v(\tau)|_p^2. \end{aligned}$$

Therefore,  $\lim_{n \rightarrow \infty} |\omega^n(\tau_n)|_p = |v(\tau)|_p$ . Since  $\omega^n(\tau_n) \rightarrow v(\tau)$  in the space  $h^{p-1}$  as  $n \rightarrow \infty$ , then the required convergence follows.  $\square$

**Lemma 3.5.** In the space  $C([-\bar{\tau}, \bar{\tau}], h^p)$ ,  $\omega^n(\cdot) \rightarrow v(\cdot)$  as  $n \rightarrow \infty$ .

*Proof:* Suppose this statement is invalid. Then there exists  $\delta > 0$  and a sequence  $\{\tau^n\}_{n \in \mathbb{N}} \subset [-\bar{\tau}, \bar{\tau}]$  such that

$$|\omega^n(\tau^n) - v(\tau^n)|_p \geq \delta.$$

Let  $\{\tau^{n_k}\}_{k \in \mathbb{N}}$  be a subsequence of the sequence  $\{\tau^n\}_{n \in \mathbb{N}}$  converging to some  $\tau^0 \in [-\bar{\tau}, \bar{\tau}]$ . But  $v(\tau^{n_k}) \rightarrow v(\tau^0)$  in  $h^p$  as  $k \rightarrow \infty$ , and using Lemma 3.4, we can get  $\omega^{n_k}(\tau^{n_k}) \rightarrow v(\tau^0)$  as  $k \rightarrow \infty$  in  $h^p$ . So we get a contradiction, and Lemma 3.5 is proved.  $\square$

If  $T \leq \bar{\tau}$ , the theorem is proved, otherwise we iterate the above procedure. This finishes the proof of Theorem 3.2.  $\square$

Let  $\mathcal{S}_v^\tau$  denote the flow determined by equations (3.1) in the space  $h^p$ , and

$$B_p^v(M) := \{v \in h^p : |v|_p \leq M\}.$$

**Theorem 3.6.** For any  $M_0 > 0$  and  $T > 0$ , there exists a constant  $C > 0$  which depends only on  $M_0$  and  $T$ , such that if  $A$  is a open subset of  $B_p^v(M_0)$ , then for  $\tau \in [0, T]$ , we have

$$e^{-C\tau} \mu(A) \leq \mu(\mathcal{S}_v^\tau(A)) \leq e^{C\tau} \mu(A).$$

*Proof:* From (1.6) we know that there is constant  $M_1$  which only depends on  $M_0$  and  $T$ , such that if  $v(0) \in B_p^v(M_0)$ , then

$$v(\tau) \in B_p^v(M_1), \quad |\tau| \leq T. \quad (3.8)$$

For any  $n \in \mathbb{N}$ , consider the measure  $\mu_n = \pi_n \circ \mu$  on the subspace  $\pi_n(h^p)$ . Since  $\mu$  is a  $\zeta_0$ -admissible Gaussian measure, by (0.8)  $\mu_n$  has the following density with respect to the Lebesgue measure:

$$b_n(v^n) := (2\pi)^{-n} \prod_{j=1}^n (2\pi j)^{1+2p} \sigma_j^{-1} \exp\left\{-\frac{1}{2} \sum_{j=1}^n \frac{j^{1+2p} |\mathbf{v}_j|^2}{\sigma_j}\right\}.$$

Let  $\mathcal{S}_n^\tau$  be the flow determined by equations (3.2) on subspace  $\pi_n(h^p)$ . For any open set  $A_n \subset \pi_n(B_p^v(M_0))$ , due to Theorem A in the appendix, we have

$$\begin{aligned} & \frac{d}{d\tau} \mu_n(\mathcal{S}_n^\tau(A_n)) \\ &= \int_{\mathcal{S}_n^\tau(A_n)} \sum_{j=1}^n \left( \frac{\partial(b_n(v^n) X_j(v^n))}{\partial v_j} + \frac{\partial(b_n(v^n) X_{-j}(v^n))}{\partial v_{-j}} \right) dv^n \\ &= \int_{\mathcal{S}_n^\tau(A_n)} \sum_{j=1}^n j^{2p+1} \left( \frac{v_j X_j + v_{-j} X_{-j}}{\sigma_j} + \frac{\partial X_j}{\partial v_j} + \frac{\partial X_{-j}}{\partial v_{-j}} \right) b_n(v^n) dv^n \\ &:= \int_{\mathcal{S}_n^\tau(A_n)} c^n(v^n) b_n(v^n) dv^n \end{aligned}$$

Since  $j^{-\zeta_0}/\sigma_j = O(1)$ , using (3.3) and the Cauchy's inequality, there exists a constant  $C$  which depends only on  $M_1$ , such that

$$|c^n(v^n)| \leq C, \quad v^n \in \pi_n(B_p^v(M_1)), \quad \forall n \in \mathbb{N}. \quad (3.9)$$

We have

$$e^{-C\tau} \mu_n(A_n) \leq \mu_n(\mathcal{S}_n^\tau(A_n)) \leq e^{C\tau} \mu_n(A_n), \quad (3.10)$$

as long as  $\mathcal{S}_n^\tau(A_n) \subset \pi_n(B_p^v(M_1))$ .

Since  $\mu_n$  converges weakly to  $\mu$ , the theorem follows from (3.8), (3.10) and Theorem 3.2 (see [12, 10]).  $\square$

#### 4. Proof of the main theorem

In this section we prove Theorem 0.2 by developing a suitable infinite-dimensional version of the Anosov scheme (see [3, 4, 5, 6]), and by studying the behavior of the regular solutions of equation (1.5) and the corresponding solutions of (0.1). We fix  $p \geq 3$ . Assume  $u(0) = u_0 \in H^p$ . So

$$\Pi_{I,\varphi}(\Psi(u_0)) = (I_0, \varphi_0) \in h_{I+}^p \times \mathbb{T}^\infty, \quad p \geq 3. \quad (4.1)$$

##### 4.1. Proof of the assertion (i)

We denote

$$B_p^I(M) = \{I \in h_{I+}^p : |I|_p \leq M\}.$$

Without loss of generality, we assume that  $\bar{T} = 1$  and  $t \geq 0$ .

Fix any  $M_0 > 0$ . Let

$$(I_0, \varphi_0) \in B_p^I(M_0) \times \mathbb{T}^\infty := \Gamma_0,$$

that is,

$$v_0 = \Psi(u_0) \in B_p^v(\sqrt{M_0}).$$

Let  $(I(t), \varphi(t))$  be a regular solution of the system (1.5) with  $(I(0), \varphi(0)) = (I_0, \varphi_0)$ . Then by (1.6), there exists  $M_1 \geq M_0$  such that

$$I(t) \in B_p^I(M_1), \quad t \in [0, \epsilon^{-1}]. \quad (4.2)$$

By the definition of the perturbation we know that

$$|F(I, \varphi)|_1 \leq C_{M_1}, \quad \forall (I, \varphi) \in B_p^I(M_1) \times \mathbb{T}^\infty, \quad (4.3)$$

where the constant  $C_{M_1}$  depends only on  $M_1$ .

We denote  $I^m = (I_1, \dots, I_m, 0, 0, \dots)$ ,  $\varphi^m = (\varphi_1, \dots, \varphi_m, 0, 0, \dots)$ , and  $W^m(I) = (W_1(I), \dots, W_m(I), 0, 0, \dots)$ , for any  $m \in \mathbb{N}$ .

Fix  $n_0 \in \mathbb{N}$ . By (2.2), for any  $\rho > 0$ , there exists  $m_0 \in \mathbb{N}$ , depending only on  $n_0$  and  $\rho$ , such that if  $m \geq m_0$ , then

$$|F_k(I, \varphi) - F_k(I^m, \varphi^m)| \leq \rho, \quad \forall (I, \varphi) \in B_p^I(M_1) \times \mathbb{T}^\infty, \quad (4.4)$$

where  $k = 1, \dots, n_0$ .

From now on, we always assume that

$$(I, \varphi) \in B_p^I(M_1) \times \mathbb{T}^\infty, \quad \text{i.e.} \quad v \in B_p^v(\sqrt{M_1}).$$

By Lemma 1.4, we have

$$\begin{aligned} |G_j(I, \varphi)| &\leq \frac{C_0(j, M_1)}{\sqrt{I_j}}, \\ \left| \frac{\partial F_k}{\partial I_j}(I, \varphi) \right| &\leq \frac{C_0(k, j, M_1)}{\sqrt{I_j}}, \\ \left| \frac{\partial F_k}{\partial \varphi_j}(I, \varphi) \right| &\leq C_0(k, j, M_1). \end{aligned} \tag{4.5}$$

From Lemma 1.3 and Lemma 2.1, we know that

$$\begin{aligned} |W_j(I) - W_j(\bar{I})| &\leq C_1(j, M_1)|I - \bar{I}|_1, \\ |\langle F_k \rangle(I) - \langle F_k \rangle(\bar{I})| &\leq C_1(k, j, M_1)|I - \bar{I}|_1. \end{aligned} \tag{4.6}$$

By (2.1) we get

$$|F_k(I^{m_0}, \varphi^{m_0}) - F_k(\bar{I}^{m_0}, \bar{\varphi}^{m_0})| \leq C_2(k, m_0, M_1)|v^{m_0} - \bar{v}^{m_0}|, \tag{4.7}$$

where  $|\cdot|$  is the maximum norm.

We denote

$$C_{M_1}^{m_0, m_0} = m_0 \cdot \max\{C_0, C_1, C_2 : 1 \leq j \leq m_0, 1 \leq k \leq n_0\}.$$

Below we define a number of sets, depending on various parameters. All of them also depend on  $m_0$  and  $n_0$ , but this dependence is not indicated. For any  $\delta > 0$ , and  $T_0 > 0$ , we define a subset  $E(\delta, T_0) \subset B_p^I(M_1)$  as the collection of all  $I \in B_p^I(M_1)$  such that for every  $\varphi \in \mathbb{T}^\infty$  and any  $T \geq T_0$ , we have

$$\left| \frac{1}{T} \int_0^T [F_k(I^{m_0}, \varphi^{m_0} + W^{m_0}(I)t) - \langle F_k \rangle(I^{m_0})] dt \right| \leq \delta, \tag{4.8}$$

for  $k = 1, \dots, n_0$ . Let  $\mathcal{S}_\epsilon^t$  be the flow generated by regular solutions of the system (1.5). We define two more groups of sets.

$$S(t) = S(t, \epsilon, \delta, T_0, I, \varphi) := \{t_1 \in [0, t] : \mathcal{S}_\epsilon^{t_1}(I, \varphi) \notin E(\delta, T_0) \times \mathbb{T}^\infty\}.$$

$$N(\tilde{T}) = N(\tilde{T}, \epsilon, \delta, T_0) := \{(I, \varphi) \in \Gamma_0 : \text{Mes}[S(\epsilon^{-1}, \epsilon, \delta, T_0, I, \varphi)] \leq \tilde{T}\}.$$

Here and below  $\text{Mes}[\cdot]$  stands for the Lebesgue measure in  $\mathbb{R}$ .

Clearly,  $E(\delta, T_0)$  is a closed subset of  $B_p^I(M_1)$  and  $S(t, \delta, T_0, I, \varphi)$  is a open subset of  $[0, t]$ . The following result is the main lemma of this work:

**Lemma 4.1.** For  $k = 1, \dots, n_0$ , the  $I_k$ -component of any regular solution of (1.5) with initial data in  $N(\tilde{T}, \epsilon, \delta, T_0)$  can be written as:

$$I_k(t) = I_k(0) + \epsilon \int_0^t \langle F_k \rangle(I(s)) ds + \Xi(t),$$

where for any  $\gamma \in (0, 1)$  the function  $|\Xi(t)|$  is bounded on  $[0, \frac{1}{\epsilon}]$  by

$$\begin{aligned} & 4\epsilon C_{M_1}^{n_0, m_0} \left\{ \left[ 2(\gamma + 2T_0 C_{M_1} \epsilon)^{1/2} \right] (T_0 + \tilde{T} + \epsilon^{-1}) \right. \\ & + \left[ \frac{T_0 C_{M_1} \epsilon}{\gamma^{1/2}} + T_0 C_{M_1} \epsilon + \left( \frac{T_0 \epsilon}{2\gamma^{1/2}} + \frac{\epsilon C_{M_1} T_0^2}{3} \right) \right] (T_0 + \tilde{T} + \epsilon^{-1}) \left. \right\} \\ & + 2\epsilon C_{M_1} \tilde{T} + 2\rho + 2\delta + 2\epsilon C_{M_1} (T_0 + \tilde{T}). \end{aligned}$$

*Proof:* For any  $(I, \varphi) \in N(\tilde{T})$ , we consider the corresponding set  $S(t)$ . It is composed of open intervals of total length less than  $\min\{\tilde{T}, t\}$ . Thus at most  $[\tilde{T}/T_0]$  of them have length greater than or equal to  $T_0$ . We denote these long intervals by  $(a_i, b_i)$ ,  $1 \leq i \leq d$ ,  $d \leq \tilde{T}/T_0$  and denote by  $C(t)$  the complement of  $\cup_{1 \leq i \leq d} (a_i, b_i)$  in  $[0, t]$ .

By (4.4), we have

$$\int_0^t F_k(I(s), \varphi(s)) dt = \int_{C(t)} F_k(I^{m_0}(s), \varphi^{m_0}(s)) ds + \xi_1(t),$$

where  $|\xi_1(t)| \leq C_{M_1} \tilde{T} + \rho t$ .

The set  $C(t)$  is composed of segments  $[b_{i-1}, a_i]$  (if necessary, we set  $b_0 = 0$ , and  $a_{d+1} = t$ ). We proceed by dividing each segment  $[b_{i-1}, a_i]$  into shorter segments by points  $t_j^i$ , where  $b_i = t_1^i < t_2^i < \dots < t_{n_i}^i = a_i$ . The points  $t_j^i$  lie outside the set  $S(t)$  and  $T_0 \leq t_{j+1}^i - t_j^i \leq 2T_0$  except for the terminal segment containing the end points  $a_i$ , which may be shorter than  $T_0$ .

This partition is constructed as follows:

- If  $a_i - b_{i-1} \leq 2T_0$ , then we keep the whole segment with no subdivisions. ( $t_1^i = b_{i-1}$ ,  $t_2^i = a_i$ ).
- If  $a_i - b_{i-1} > 2T_0$ , we divide the segment in the following way:
  - a) If  $b_{i-1} + 2T_0$  does not belong to  $S(t)$ , we chose  $t_2^i = b_{i-1} + 2T_0$ , and continue by subdividing  $[t_2^i, a_i]$ ;
  - b) if  $b_{i-1} + 2T_0$  belongs to  $S(t)$ , then there are points in  $[b_{i-1} + T_0, b_{i-1} + 2T_0]$  which do not, by definition of  $b_{i-1}$ . We set  $t_2^i$  equal to one of these points and continue by subdividing  $[t_2^i, a_i]$ .

We will adopt the notation:  $h_j^i = t_{j+1}^i - t_j^i$  and  $s(i, j) = [t_j^i, t_{j+1}^i]$ . So

$$C(t) = \bigcup_{i=1}^d \bigcup_{j=1}^{n_i-1} s(i, j), \quad T_0 \leq h_j^i = |s(i, j)| \leq 2T_0, \quad j \leq n_i - 2.$$

By its definition,  $C(t)$  contains at most  $[\tilde{T}/T_0] + 1$  segments  $[b_{i-1}, a_i]$ , thus  $C(t)$  contains at most  $[\tilde{T}/T_0] + 1$  terminal subsegments of length less than  $T_0$ . Since all other segments have length no less than  $T_0$  and  $t \leq \frac{1}{\epsilon}$ , the number of these segments is not greater than  $[\epsilon T_0]^{-1}$ . So the total number of subsegments  $s(i, j)$  is bounded by  $1 + [(\epsilon T_0)^{-1}] + [\tilde{T}/T_0]$ .

For each segment  $s(i, j)$  we define a subset  $\Lambda(i, j)$  of  $\{1, 2, \dots, m_0\}$  in the following way:

$$l \in \Lambda(i, j) \iff \exists t \in s(i, j), \quad I_l(t) < \gamma.$$

If  $l \in \Lambda$ , then by (4.3) we have

$$|I_l(t)| < 2T_0 C_{M_1} \epsilon + \gamma, \quad t \in s(i, j). \quad (4.9)$$

For  $I = (I_1, I_2, \dots)$  and  $\varphi = (\varphi_1, \varphi_2, \dots)$  we set

$$\lambda_{i,j}(I) = \hat{I}, \quad \lambda_{i,j}(\varphi) = \hat{\varphi},$$

where  $\hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2, \dots)$  and  $\hat{I} = (\hat{I}_1, \hat{I}_2, \dots)$  are defined by the following relation:

$$\text{If } l \in \Lambda(i, j), \quad \text{then } \hat{I}_l = 0, \quad \hat{\varphi}_l = 0, \quad \text{else } \hat{I}_l = I_l, \quad \hat{\varphi}_l = \varphi_l.$$

We also denote  $\lambda_{i,j}(I, \varphi) = (\lambda_{i,j}(I), \lambda_{i,j}(\varphi))$  and when the segment  $s(i, j)$  is clearly indicated, we write for short  $\lambda_{i,j}(I, \varphi) = (\hat{I}, \hat{\varphi})$ .

Then on  $s(i, j)$ , using (4.7) and (4.9) we obtain

$$\begin{aligned} & \int_{s(i,j)} \left| F_k \left( I^{m_0}(s), \varphi^{m_0}(s) \right) - F_k \left( \lambda_{i,j}(I^{m_0}(s), \varphi^{m_0}(s)) \right) \right| ds \\ & \leq \int_{s(i,j)} C_{M_1}^{m_0, m_0} \left| I^{m_0}(s) - \lambda_{i,j}(I^{m_0}(s)) \right|^{1/2} ds \\ & \leq 2T_0 C_{M_1}^{m_0, m_0} (\gamma + 2T_0 C_{M_1} \epsilon)^{1/2}. \end{aligned} \quad (4.10)$$

In Proposition 1-5 below,  $k = 1, \dots, n_0$ .

**Proposition 1.**

$$\int_{C(t)} F_k \left( I^{m_0}(s), \varphi^{m_0}(s) \right) ds = \sum_{i,j} \int_{s(i,j)} F_k \left( I^{m_0}(t_j^i), \varphi^{m_0}(s) \right) ds + \xi_2(t),$$

where

$$|\xi_2| \leq 4C_{M_1}^{m_0, m_0} \left[ (\gamma + 2T_0 C_{M_1} \epsilon)^{1/2} + \gamma^{-1/2} T_0 C_{M_1} \epsilon \right] (T_0 + \tilde{T} + \epsilon^{-1}). \quad (4.11)$$

*Proof:* We may write  $\xi_2(t)$  as

$$\begin{aligned} \xi_2(t) &= \sum_{i,j} \int_{s(i,j)} \left[ F_k \left( I^{m_0}(s), \varphi^{m_0}(s) \right) - F_k \left( I^{m_0}(t_j^i), \varphi^{m_0}(s) \right) \right] ds \\ &:= \sum_{i,j} I(i, j). \end{aligned}$$

For each  $s(i, j)$ , we have

$$\begin{aligned} & \int_{s(i,j)} \left| F_k \left( \hat{I}^{m_0}(s), \hat{\varphi}^{m_0}(s) \right) - F_k \left( \hat{I}^{m_0}(t_j^i), \hat{\varphi}^{m_0}(s) \right) \right| ds \\ & \leq \int_{s(i,j)} \gamma^{-1/2} C_{M_1}^{m_0, m_0} \left| \hat{I}^{m_0}(s) - \hat{I}^{m_0}(t_j^i) \right| ds \\ & \leq 2\gamma^{-1/2} T_0^2 C_{M_1} \epsilon. \end{aligned} \quad (4.12)$$

We replace the integrand  $F_k(I^{m_0}, \varphi^{m_0})$  by  $F_k(\hat{I}^{m_0}, \hat{\varphi}^{m_0})$ . Using (4.10) and (4.12) we obtain that

$$I(i, j) \leq 4T_0 C_{M_1}^{m_0, m_0} \left[ (\gamma + 2T_0 C_{M_1} \epsilon)^{1/2} + \gamma^{-1/2} T_0 C_{M_1} \epsilon \right].$$

The inequality (4.11) follows.  $\square$

On each subsegment  $s(i, j)$ , we now consider the unperturbed linear dynamics  $\varphi_j^i(t)$  of the angles  $\varphi^{m_0} \in \mathbb{T}^{m_0}$  :

$$\varphi_j^i(t) = \varphi^{m_0}(t_j^i) + W^{m_0}(I(t_j^i))(t - t_j^i) \in \mathbb{T}^{m_0}, \quad t \in s(i, j).$$

**Proposition 2.**

$$\sum_{i,j} \int_{s(i,j)} F_k \left( I^{m_0}(t_j^i), \varphi^{m_0}(s) \right) ds = \sum_{i,j} \int_{s(i,j)} F_k \left( I^{m_0}(t_j^i), \varphi_j^i(s) \right) ds + \xi_3(t),$$

where

$$\begin{aligned} |\xi_3(t)| &\leq 4C_{M_1}^{m_0, m_0} (\gamma + 2T_0 C_{M_1} \epsilon)^{1/2} (T_0 + \tilde{T} + \epsilon^{-1}) \\ &\quad + (C_{M_1}^{m_0, m_0})^2 \left( \frac{2T_0 \epsilon}{\gamma} + \frac{4\epsilon C_{M_1} T_0^2}{3} \right) (T_0 + \tilde{T} + \epsilon^{-1}). \end{aligned} \quad (4.13)$$

*Proof.* For each  $s(i, j)$  we have

$$\begin{aligned} &\int_{s(i,j)} \left| \lambda_{i,j} \left( \varphi^{m_0}(s) - \varphi_j^i(s) \right) \right| ds \\ &\leq \int_{s(i,j)} \int_{t_j^i}^s \left| \lambda_{i,j} \left( \epsilon G^{m_0}(I(s'), \varphi(s')) + W^{m_0}(I(s')) - W^{m_0}(I(t_j^i)) \right) \right| ds' ds \\ &\leq \int_{s(i,j)} \int_{t_j^i}^s C_{M_1}^{m_0, m_0} \left[ \epsilon \gamma^{-1/2} + |I(s') - I(t_j^i)|_1 \right] ds' ds \\ &\leq \int_{s(i,j)} C_{M_1}^{m_0, m_0} \left[ \gamma^{-1/2} \epsilon (s - t_j^i) + \frac{1}{2} C_{M_1} \epsilon (s - t_j^i)^2 \right] ds \\ &\leq C_{M_1}^{m_0, m_0} \left( \frac{2T_0^2 \epsilon}{\sqrt{\gamma}} + \frac{4\epsilon C_{M_1} T_0^3}{3} \right). \end{aligned}$$

Here the first inequality comes from equation (1.4), and using (4.5) and (4.6) we can get the second inequality. The third one follows from (4.3).

Using again (4.5), we get

$$\begin{aligned} &\int_{s(i,j)} \left[ F_k \left( \lambda_{i,j}(I^{m_0}(t_j^i), \varphi^{m_0}(s)) \right) - F_k \left( \lambda_{i,j}(I^{m_0}(t_j^i), \varphi_j^i(s)) \right) \right] ds \\ &\leq \int_{s(i,j)} C_{M_1}^{m_0, m_0} \left| \lambda_{i,j} \left( \varphi^{m_0}(s) - \varphi_j^i(s) \right) \right| ds \\ &\leq (C_{M_1}^{m_0, m_0})^2 \left( \frac{2T_0^2 \epsilon}{\sqrt{\gamma}} + \frac{4\epsilon C_{M_1} T_0^3}{3} \right). \end{aligned}$$

Therefore (4.13) holds for the same reason as (4.11).  $\square$

We will now compare the integral  $\int_{s(i,j)} F_k(I^{m_0}(t_j^i), \varphi_j^i(s)) ds$  with the average value  $\langle F_k(I^{m_0}(t_j^i)) \rangle h_j^i$ .



**Proposition 3.**

$$\sum_{i,j} \int_{s(i,j)} F_k \left( I^{m_0}(t_j^i), \varphi_j^i(s) \right) ds = \sum_{i,j} h_j^i \langle F_k \rangle \left( I^{m_0}(t_j^i) \right) + \xi_4(t),$$

where

$$|\xi_4(t)| \leq \frac{2\delta}{\epsilon} + 2C_{M_1}(T_0 + \tilde{T}). \quad (4.14)$$

*Proof:* We divide the set of segments  $s(i, j)$  into two subsets  $\Delta_1$  and  $\Delta_2$ . Namely,  $s(i, j) \in \Delta_1$  if  $h_j^i \geq T_0$  and  $s(i, j) \in \Delta_2$  otherwise.

(i)  $s(i, j) \in \Delta_1$ . In this case, by (4.8), we have

$$\left| \int_{s(i,j)} \left[ F_k \left( I^{m_0}(t_j^i), \varphi_j^i(s) \right) - \langle F_k \rangle \left( I^{m_0}(t_j^i) \right) \right] ds \right| \leq \delta h_j^i.$$

So

$$\sum_{s(i,j) \in \Delta_1} \left| \int_{s(i,j)} F_k \left( I^{m_0}(t_j^i), \varphi_j^i(s) \right) ds - \langle F_k \rangle \left( I^{m_0}(t_j^i) \right) h_j^i \right| \leq \delta \sum_{s(i,j) \in \Delta_1} h_j^i \leq \frac{2\delta}{\epsilon}.$$

(ii)  $s(i, j) \in \Delta_2$ . Now, using (4.3) we get

$$\left| \int_{s(i,j)} F_k \left( I^{m_0}(t_j^i), \varphi_j^i(s) \right) ds - \langle F_k \rangle \left( I^{m_0}(t_j^i) \right) h_j^i \right| \leq 2C_{M_1} h_j^i \leq 2C_{M_1} T_0.$$

Since  $\text{Card}(\Delta_2) \leq (1 + \tilde{T}/T_0)$ , then

$$\sum_{s(i,j) \in \Delta_2} \left| \int_{s(i,j)} F \left( I^{m_0}(t_j^i), \varphi_j^i(s) \right) ds - \langle F_k \rangle \left( I^{m_0}(t_j^i) \right) h_j^i \right| \leq 2C_{M_1}(\tilde{T} + T_0).$$

This implies the inequality (4.14).  $\square$

**Proposition 4.**

$$\sum_{i,j} h_j^i \langle F_k \rangle \left( I^{m_0}(t_j^i) \right) = \int_{C(t)} \langle F_k \rangle \left( I^{m_0}(s) \right) ds + \xi_5(t),$$

where

$$|\xi_5(t)| \leq 4\epsilon C_{M_1} C_{M_1}^{m_0, m_0} T_0 (T_0 + \tilde{T} + \epsilon^{-1}). \quad (4.15)$$

*Proof:* Indeed, as

$$|\xi_5(t)| = \left| \sum_{i,j} \int_{s(i,j)} \left[ \langle F_k \rangle (I^{m_0}(s)) - \langle F_k \rangle (I^{m_0}(t_j^i)) \right] ds \right|,$$

using (4.3) and (4.6) we get

$$\begin{aligned} |\xi_5(t)| &\leq \sum_{i,j} \int_{s(i,j)} C_{M_1}^{n_0, m_0} |I^{m_0}(s) - I^{m_0}(t_j^i)| ds \\ &\leq \epsilon \sum_{i,j} C_{M_1} C_{M_1}^{m_0, m_0} (h_j^i)^2 \leq 4\epsilon C_{M_1} C_{M_1}^{m_0, m_0} T_0 (T_0 + \tilde{T} + \epsilon^{-1}). \quad \square \end{aligned}$$

Finally,

**Proposition 5.**

$$\int_{C(t)} \langle F_k \rangle (I^{m_0}(s)) ds = \int_0^t \langle F_k \rangle (I(s)) ds + \xi_6(t),$$

and  $|\xi_6(t)|$  is bounded by  $C_{M_1}\tilde{T} + \rho t$ .  $\square$

Gathering the estimates in Propositions 1-5, we obtain

$$\begin{aligned} I_k(t) &= I_k(0) + \epsilon \int_0^t F_k(I(s), \varphi(s)) ds \\ &= I_k(0) + \epsilon \int_0^t \langle F_k \rangle (I(s)) ds + \Xi(t), \end{aligned}$$

where

$$\begin{aligned} |\Xi(t)| &\leq \epsilon \sum_{i=1}^6 |\xi_i(t)| \\ &\leq 4\epsilon C_{M_1}^{m_0, m_0} \left[ 2(\gamma + 2T_0 C_{M_1} \epsilon)^{1/2} + \frac{T_0 C_{M_1} \epsilon}{\gamma^{1/2}} + T_0 C_{M_1} \epsilon \right. \\ &\quad \left. + \left( \frac{T_0 \epsilon}{2\gamma^{1/2}} + \frac{\epsilon C_{M_1} T_0^2}{3} \right) \right] (T_0 + \tilde{T} + \epsilon^{-1}) + 2\epsilon C_{M_1} T_1 \\ &\quad + 2\rho + 2\delta + 2\epsilon C_{M_1} (T_0 + \tilde{T}), \quad t \in [0, \frac{1}{\epsilon}]. \end{aligned}$$

Lemma 4.1 is proved.  $\square$

**Corollary 4.2.** For any  $\bar{\rho} > 0$ , with a suitable choice of  $\rho, \gamma, \delta, T_0, \tilde{T}$ , the function  $|\Xi(t)|$  in Lemma 4.1 can be made smaller than  $\bar{\rho}$ , if  $\epsilon$  is small enough.

*Proof:* We choose

$$\gamma = \epsilon^\alpha, \quad T_0 = \epsilon^{-\sigma}, \quad \tilde{T} = \frac{\bar{\rho}}{9C_{M_1}\epsilon}, \quad \delta = \rho = \frac{\bar{\rho}}{9}$$

with

$$1 - \frac{\alpha}{2} - \sigma > 0, \quad 0 < \sigma < \frac{1}{2}.$$

Then for  $\epsilon$  sufficiently small we have

$$|\Xi(t)| < \bar{\rho}. \quad \square$$

On the Hilbert space  $h^p$ , we adopt a  $\zeta_0$ -admissible Gaussian measure  $\mu$ . Define corresponding measures  $\mu_I = \Pi_I \circ \mu$  and  $\mu_{I,\varphi} = \Pi_{I,\varphi} \circ \mu$  in the spaces  $h_{I+}^p$  and  $h_{I+}^p \times \mathbb{T}^\infty$ .

**Lemma 4.3.** The measure  $\mu_{I,\varphi}$  is a product measure  $d\mu_{I,\varphi} = d\mu_I d\varphi$ , where  $d\varphi$  is the Haar measure on  $\mathbb{T}^\infty$ .

*Proof:* Since the measure  $\mu$  is invariant under rotations  $\Phi_\theta$ , the  $\Pi_\varphi \circ d\mu$  is a measure on  $\mathbb{T}^\infty$ , invariant under the rotations. So this is the Haar measure  $d\varphi$ . Consequently the image of the measure  $\mu_{I,\varphi}$  under the natural projection  $(I, \varphi) \mapsto \varphi$  is  $d\varphi$ . Since the spaces  $h_{I+}^p$  and  $\mathbb{T}^\infty$  are separable, then for  $\varphi \in \mathbb{T}^\infty$  there exists a Borel probability

measure  $\pi_\varphi(dI)$  on  $h_{I+}^p$  such that  $\mu_{I,\varphi} = \pi_\varphi(dI)d\varphi$ . That is, for any bounded continuous function  $f(I, \varphi)$ , we have

$$\langle \mu_{I,\varphi}, f \rangle = \int_{\mathbb{T}^\infty} \left( \int_{h_{I+}^p} f(I, \varphi) \pi_\varphi(dI) \right) d\varphi.$$

(see e.g. [9]). For any  $\theta \in \mathbb{T}^\infty$  we have

$$\begin{aligned} \langle \mu_{I,\varphi}, f \rangle &= \langle \mu_{I,\varphi}, f \circ \Phi_\theta \rangle \\ &= \int \int f(I, \varphi + \theta) \pi_\varphi(dI) d\varphi = \int \int f(I, \varphi) \pi_{\varphi-\theta}(dI) d\varphi. \end{aligned}$$

Integrating in  $d\theta$  we see that

$$\mu_{I,\varphi}(dId\varphi) = d\mu'(dI)d\varphi,$$

where  $d\mu'(dI) = \int_{\mathbb{T}^\infty} \pi_\theta(dI) d\varphi$ . We must have  $d\mu' = d\mu_I$ , and the assertion of the lemma is proved.  $\square$

The two lemmas below deal with the sets  $E$  and  $N$ , defined at the beginning of this section.

**Lemma 4.4.** For any  $\delta > 0$ ,  $\lim_{T_0 \rightarrow \infty} \mu_I(B_p^I(M_1) \setminus E(\delta, T_0)) = 0$ .

*Proof:* From the definition of  $E(\delta, T_0)$ , we know that

$$E(\delta, T_0) \subset E(\delta, T'_0), \quad \text{if } T_0 \leq T'_0.$$

Let  $E_\infty(\delta) := \bigcup_{T_0 > 0} E(\delta, T_0)$ . Due to the inclusion above we have to check that

$$\mu_I(B_p^I(M_1) \setminus E_\infty(\delta)) = 0.$$

Denote

$$\mathcal{R}(N) := \bigcup_{L \in \mathbb{Z}^{m_0} \setminus \{0\}, |L| \leq N} \{I \in B_p^I(M_1) : W^{m_0}(I) \cdot L = 0\},$$

where  $W^{m_0}(I) = (W_1(I), \dots, W_{m_0}(I))$ . Let us write  $F_k(I^{m_0}, \varphi^{m_0})$  as a Fourier series  $F_k(I^{m_0}, \varphi^{m_0}) = \sum_{L \in \mathbb{Z}^{m_0}} F_k^L e^{iL \cdot \varphi^{m_0}}$ , where  $F_k^L = F_k^L(I^{m_0})$ . Then there exists  $N_0 > 0$  such that

$$\left| F_k(I^{m_0}, \varphi^{m_0}) - \sum_{|L| \leq N_0} F_k^L e^{iL \cdot \varphi^{m_0}} \right| < \frac{\delta}{2}, \quad k = 1, \dots, n_0.$$

Arguing as in the proof of Lemma 2.2, we see that if  $I \notin \mathcal{R}(N_0)$ , then

$$\left| \sum_{0 \neq |L| \leq N_0} \frac{1}{T_0} \int_0^{T_0} F_k^L e^{iL \cdot W^{m_0} t} dt \right| \leq \frac{2}{T_0} \left( \inf_{0 \neq |L| \leq N_0} |L \cdot W^{m_0}| \right)^{-1} \sum_{|L| \leq N_0} |F_k^L|.$$

where  $W^{m_0} = W^{m_0}(I)$ . The r.h.s of the above inequality can be made smaller than  $\delta/2$  by choosing  $T_0$  large enough. So we have

$$B_p^I(M_1) \setminus \mathcal{R}(N_0) \subset E_\infty(\delta),$$

and it remains to show that

$$\mu_I(\mathcal{R}(N_0)) = 0.$$

By Lemma 1.2,

$$W^{m_0}(I) \cdot L \not\equiv 0, \quad \forall L \in \mathbb{Z}^{m_0} \setminus \{0\},$$

Since  $W^{m_0}(I)$  is analytic with respect to  $I$  and  $\mu_I$  is a non-degenerated Gaussian measure, then due to Theorem 1.6 in [13], for any  $L \in \mathbb{Z}^{m_0}$ , we have

$$\mu_I(\{I \in h_I^p : W^{m_0}(I) \cdot L = 0\}) = 0.$$

Therefore,

$$\mu_I(\mathcal{R}(N_0)) = 0. \quad \square$$

**Lemma 4.5.** Fix any  $\delta > 0$ ,  $\bar{\rho} > 0$ . Then for every  $\nu > 0$  we can find  $T_0 > 0$  such that

$$\mu_{I,\varphi}(\Gamma_0 \setminus N) < \nu,$$

where  $N = N(\frac{\bar{\rho}}{9C_{M_1}\epsilon}, \epsilon, \delta, T_0)$ .

*Proof:* Let us denote  $\Gamma_E = E(\delta, T_0) \times \mathbb{T}^\infty$ ,  $\Gamma_1 = B_p^I(M_1) \times \mathbb{T}^\infty$  and  $\Gamma_E^\infty := \bigcup_{T_0 > 0} \Gamma_E(\delta, T_0)$ . Since the sets  $\Gamma_E(\delta, T_0)$  are increasing with  $T_0$ , then from Lemmas 4.3 and 4.4 we know that

$$\lim_{T_0 \rightarrow \infty} \mu_{I,\varphi}(\Gamma_1 \setminus \Gamma_E(\delta, T_0)) = \mu_{I,\varphi}(\Gamma_1 \setminus \Gamma_E^\infty) = 0. \quad (4.16)$$

Let  $d\mu_1$  be the measure  $d\mu dt$  on  $h^p \times \mathbb{R}$ , and  $\mathcal{S}_{v,\epsilon}^t$  be the flow of the perturbed KdV equation (1.2) on  $h^p$ . We now define following subset of  $h^p \times \mathbb{R}$ :

$$B' = \left\{ (v, t) : \mathcal{S}_{v,\epsilon}^t(v) \in \Pi_{I,\varphi}^{-1}(\Gamma_1 \setminus \Gamma_E(\delta, T_0)), v \in B_p^v(\sqrt{M_0}), t \in [0, \frac{1}{\epsilon}] \right\}.$$

By Theorem 3.6, there exists a constant  $C_2(M_1)$  depending only on  $M_1$  such that

$$\begin{aligned} \mu_1(B') &= \int_0^{\epsilon^{-1}} \mu \left( \mathcal{S}_{v,\epsilon}^{-t} \left( \Pi_{I,\varphi}^{-1}(\Gamma_1 \setminus \Gamma_E(\delta, T_0)) \right) \cap \Pi_{I,\varphi}^{-1}(\Gamma_0) \right) dt \\ &\leq \frac{1}{\epsilon} e^{C_2(M_1)} \mu \left( \Pi_{I,\varphi}^{-1}(\Gamma_1 \setminus \Gamma_E(\delta, T_0)) \right) \\ &= \frac{1}{\epsilon} e^{C_2(M_1)} \mu_{I,\varphi}(\Gamma_1 \setminus \Gamma_E(\delta, T_0)). \end{aligned}$$

For  $v \in \Pi_{I,\varphi}^{-1}(\Gamma_0)$ , we define

$$S(I, \varphi) = S(v) = \{t \in [0, \epsilon^{-1}] : \mathcal{S}_{v,\epsilon}^t(v) \in B_p^v(\sqrt{M_1}) \setminus \Pi_{I,\varphi}^{-1}(\Gamma_E(\delta, T_0))\}.$$

By the Fubini theorem, we have

$$\mu_1(B') = \int_{\Pi_I^{-1}(\Gamma_0)} \text{Mes}(S(v)) \mu(dv),$$

Thus

$$\begin{aligned} \mu_{I,\varphi}(\Gamma_0 \setminus N) &= \mu_{I,\varphi} \left( \{(I, \varphi) \in \Gamma_0 : \text{Mes}(S(I, \varphi)) > \frac{\bar{\rho}}{9C_{M_1}\epsilon}\} \right) \\ &\leq \frac{9C_{M_1} e^{C_2(M_1)}}{\bar{\rho}} \mu_{I,\varphi}(\Gamma_1 \setminus \Gamma_E(\delta, T_0)), \end{aligned}$$

by the Chebyshev inequality. In view of (4.16) the term on the right hand side becomes arbitrary small when  $T_0$  is large enough. The statement of Lemma 4.5 follows.  $\square$

We pass to the slow time  $\tau = \epsilon t$ . Let  $v^\epsilon(\tau)$ ,  $\tau \in [0, 1]$ , be a solution of the equation (3.1) and  $(I^\epsilon(\tau), \varphi^\epsilon(\tau)) = \Pi_{I, \varphi}(v^\epsilon(\tau))$ .

By Lemma 2.1 and (3.3), we know that for any  $p \geq 0$ , the mapping

$$F_J : h_I^p \rightarrow h_I^{p+\zeta_0}, \quad J \mapsto \langle F \rangle(J),$$

where  $\langle F \rangle(J) = (\langle F_1 \rangle(J), \langle F_2 \rangle(J), \dots)$  is analytic. Hence, there exists  $C_3(M_1)$  such that

$$|F_J(J_1) - F_J(J_2)|_p \leq C_3(M_1)|J_1 - J_2|_p, \quad J_1, J_2 \in B_p^I(2M_1). \quad (4.17)$$

Using Picard's theorem, for any  $J_0 \in B_p^I(M_1)$  there exists a unique solution  $J(t)$  of the averaged equation (0.7) with  $J(0) = J_0$ . We denote

$$T(J_0) := \inf\{\tau > 0 : |J(\tau)|_p > 2M_1\}.$$

Now we are in a position to prove the assertion (i) of Theorem 0.2.

For any  $\bar{\rho} > 0$ , there exist  $n_1$  such that

$$\begin{aligned} |F(I, \varphi) - F^{n_1}(I, \varphi)|_p &< \frac{\bar{\rho}}{8} e^{-C_3(M_1)}, \quad (I, \varphi) \in B_p^I(2M_1) \times \mathbb{T}^\infty, \\ |\langle F \rangle(J) - \langle F \rangle^{n_1}(J)|_p &< \frac{\bar{\rho}}{8} e^{-C_3(M_1)}, \quad J \in B_p^I(2M_1). \end{aligned} \quad (4.18)$$

Choose  $\rho_0$  such that

$$8 \sum_{j=1}^{n_1} j^{1+2p} \rho_0 = \bar{\rho} e^{-C_3(M_1)}.$$

By Lemmata 4.1 and 4.2, there is a set  $\Gamma_{\bar{\rho}} = N(\frac{\rho_0}{9C_{M_1}\epsilon}, \epsilon, \frac{\rho_0}{9}, \epsilon^{-\sigma})$ ,  $\sigma < 1/2$ , such that if  $\epsilon$  is small enough and  $(I^\epsilon(0), \varphi^\epsilon(0)) \in \Gamma_{\bar{\rho}}$ , then

$$I_k(\tau) = I_k(0) + \int_0^\tau \langle F_k \rangle(I(s)) ds + \xi_k(\tau), \quad |\xi_k(\tau)| < \rho_0, \quad \tau \in [0, 1],$$

for  $k = 1, \dots, n_1$ . Therefore, by (4.17) and (4.18),

$$|I(\tau) - J(\tau)|_p \leq \int_0^\tau C_3(M_1)|I(\tau) - J(\tau)|_p ds + \xi_0(\tau), \quad |\xi_0(\tau)| \leq \frac{\bar{\rho}}{2} e^{-C_3(M_1)},$$

for  $(I(0), \varphi(0)) \in \Gamma_{\bar{\rho}}$ ,  $I(0) = J(0)$  and  $|\tau| \leq \min\{1, T(J(0))\}$ . By Gronwall's lemma,

$$|I(\tau) - J(\tau)|_p \leq \bar{\rho}, \quad |\tau| \leq \min\{1, T(J(0))\}.$$

Assuming that  $\bar{\rho} \ll M_1$ , we get from the definition of  $T(J(0))$  that  $T(J(0))$  is bigger than 1. This establishes inequality (0.9). From Lemma 4.5 we know that  $\lim_{\epsilon \rightarrow 0} \mu_{I, \varphi}(\Gamma_0 - \Gamma_{\bar{\rho}}) = 0$ .  $\square$

#### 4.2. Proof of the assertion (ii)

It is not hard to see that the assertion for any  $0 \leq \bar{T}_1 < \bar{T}_2 \leq 1$  would follow if we can prove it for  $\bar{T}_1 = 0$ ,  $\bar{T}_2 = 1$ . So we assume that  $\bar{T}_1 = 0$ , and  $\bar{T}_2 = 1$ . For any  $(m, n) \in \mathbb{N}^2$ , we fix  $\alpha < 1/8$ , and denote

$$\begin{aligned} \mathcal{B}_m(\epsilon) &:= \left\{ I \in B_p^I(M_1) : \inf_{k \leq m} |I_k| < \epsilon^\alpha \right\}, \\ \mathcal{R}_{m,n}(\epsilon) &:= \bigcup_{|L| \leq n, L \in \mathbb{Z}^m \setminus \{0\}} \left\{ I \in B_p^I(M_1) : |W(I) \cdot L| < \epsilon^\alpha \right\}. \end{aligned}$$

Then let

$$\Upsilon_{m,n}(\epsilon) = \left( \bigcup_{m_0 \leq m} \mathcal{R}_{m_0,n}(\epsilon) \right) \cup \mathcal{B}_m(\epsilon).$$

Denote

$$S(\epsilon, m, n, I_0, \varphi_0) = \{\tau \in [0, 1] : I^\epsilon(\tau) \in \Upsilon_{m,n}(\epsilon)\}$$

and fix any  $\nu > 0$ . Then using Theorem 3.6 and arguing as in Lemma 4.4 and Lemma 4.5, we get that, for any  $(m, n) \in \mathbb{N}^2$ , there exists open subset  $\Gamma_\nu^{m,n} \subset \Gamma_0$ ,  $\epsilon_{m,n} > 0$  and a positive function  $\rho_{m,n}(\epsilon)$ , converging to zero as  $\epsilon \rightarrow 0$ , such that

$$\mu_{I,\varphi}(\Gamma_0 - \Gamma_\nu^{m,n}) < \frac{\nu}{2^{mn}} \quad \text{and} \quad \text{Mes}(S(\epsilon, m, n, I_0, \varphi_0)) \leq \rho_{m,n}(\epsilon),$$

if  $(I_0, \varphi_0) \in \Gamma_\nu^{m,n}$  and  $\epsilon \leq \epsilon_{m,n}$ . Let

$$\Gamma_\nu = \bigcap_{(m,n) \in \mathbb{N}^2} \Gamma_\nu^{m,n},$$

then

$$\mu_{I,\varphi}(\Gamma_0 - \Gamma_\nu) < \nu. \tag{4.19}$$

The sets  $\Gamma_\nu$  may be chosen in such a manner that

$$\Gamma_{\nu_1} \subset \Gamma_{\nu_2}, \quad \text{if } \nu_2 < \nu_1. \tag{4.20}$$

For any  $(I_0, \varphi_0) \in \Gamma_\nu$ , consider a solution  $(I^\epsilon(\tau), \varphi^\epsilon(\tau))$  such that

$$(I^\epsilon(0), \varphi^\epsilon(0)) = (I_0, \varphi_0).$$

Fix  $m \in \mathbb{N}$ , take a bounded Lipschitz function  $g$  defined on the torus  $\mathbb{T}^m \subset \mathbb{T}^\infty$  such that  $\text{Lip}(g) \leq 1$  and  $|g|_{L^\infty} \leq 1$ . Let  $\sum_{s \in \mathbb{Z}^m} g_s e^{is \cdot \varphi}$  be its Fourier series. Then for any  $\rho > 0$ , there exists  $n$ , such that if we denote  $\bar{g}_n = \sum_{|s| \leq n} g_s e^{is \cdot \varphi}$ , then

$$\left| g(\varphi) - \bar{g}_n(\varphi) \right| < \frac{\rho}{2}, \quad \forall \varphi \in \mathbb{T}^m.$$

For any  $(I_0, \varphi_0) \in \Gamma_\nu$ , we consider the set  $S(\epsilon, m, n, I_0, \varphi_0)$ . It is composed of open intervals of total length less than  $\tilde{T} = \rho_{m,n}(\epsilon)$ . Proceeding as in Lemma 4.1 and Corollary 4.2, we find that for  $\epsilon$  small enough we have

$$\left| \int_0^1 g(\varphi^{\epsilon,m}(\tau)) d\tau - \int_{\mathbb{T}^m} g(\varphi) d\varphi \right| < \rho.$$

That is ,

$$\left| \int g(\varphi) \mu_{\tilde{T}_1, \tilde{T}_2}^\epsilon(d\varphi) - \int g(\varphi) d\varphi \right| \longrightarrow 0 \quad \text{as } \epsilon \rightarrow 0, \tag{4.21}$$

for any Lipschitz function as above. Hence,  $\mu_{\tilde{T}_1, \tilde{T}_2}^\epsilon$  converges weakly to  $d\varphi$  (see [9]). This proves the required assertion with  $\Gamma_\varphi$  replaced by  $\Gamma_\nu$ . Let us choose

$$\Gamma_\varphi = \bigcup_{\nu > 0} \Gamma_\nu.$$

Then

$$\mu_{I,\varphi}(\Gamma_0 - \Gamma_\varphi) = 0,$$

by (4.19) and (4.20), and for any  $(I_0, \varphi_0) \in \Gamma_\varphi$  the required convergence of measures holds. This proves the second assertion of Theorem 0.2.  $\square$

### 5. Application to a special case

In this section we prove Proposition 0.3. Clearly, we only need to prove the statement (ii) of assumption A. Let  $\mathcal{F} : H^m \rightarrow \mathbb{R}$  be a smooth functional (for some  $m \geq 0$ ). If  $u(t)$  is a solution of (0.1), then

$$\frac{d}{dt}\mathcal{F}(u(t)) = \langle \nabla \mathcal{F}(u(t)), -V(u) + \epsilon f(x) \rangle.$$

In particular, if  $\mathcal{F}(u)$  is an integral of motion for the KdV equation, then we have  $\langle \nabla \mathcal{F}(u(t)), V(u) \rangle = 0$ , so

$$\frac{d}{dt}\mathcal{F}(u(t)) = \epsilon \langle \nabla \mathcal{F}(u(t)), f(x) \rangle.$$

Since  $\|u(0)\|_0^2$  is an integral of motion, then

$$\frac{d}{dt}\|u(t)\|_0^2 = 2\epsilon \langle u, f(x) \rangle \leq \epsilon(\|u\|_0^2 + \|f(x)\|_0^2).$$

Thus we have

$$\|u(t)\|_0^2 \leq e^{\epsilon t}(\|u(0)\|_0^2 + \epsilon t\|f(x)\|_0^2). \quad (5.1)$$

The KdV equation has infinitively many integral of motion  $\mathcal{J}_m(u)$ ,  $m \geq 0$ . The integral  $\mathcal{J}_m$  can be written as

$$\mathcal{J}_m(u) = \|u\|_m^2 + \sum_{r=3}^m \sum_{\mathbf{m}} \int C_{r,\mathbf{m}} u^{(m_1)} \dots u^{(m_r)} dx,$$

where the inner sum is taken over all integer  $r$ -vectors  $\mathbf{m} = (m_1, \dots, m_r)$ , such that  $0 \leq m_j \leq m-1$ ,  $j = 1, \dots, r$  and  $m_1 + \dots + m_r = 4 + 2m - 2r$ . Particularly,  $\mathcal{J}_0(u) = \|u\|_0^2$ .

Lets consider

$$I = \int u^{(m_1)} \dots f^{(m_i)} \dots u^{(m_{r_1})} dx, \quad m_1 + \dots + m_{r_1} = M,$$

where  $r_1 \geq 2$ ,  $M \geq 1$ , and  $0 \leq m_j \leq \mu - 1$ . Then, by Hölder's inequality,

$$|I| \leq \|u^{(m_1)}\|_{L_{p_1}} \dots \|f(x)\|_{L_{p_i}} \dots \|u^{(m_{r_1})}\|_{L_{p_f}}, \quad p_j = \frac{M}{m_j} \leq \infty.$$

Applying next the Gagliardo-Nirenberg and the Young inequalities, we obtain that

$$|I| \leq \delta \|u\|_\mu^2 + C_\delta \|u\|_0^{C_1}, \quad \forall \delta > 0, \quad (5.2)$$

where  $C_\delta$  and  $C_1$  do not depend on  $u$ . Below we denote  $C$  a positive constant independent of  $u$ , not necessary the same in each inequality. Let

$$I_1 := \langle \nabla \mathcal{J}_m(u), f \rangle = \langle u^{(m)}, f^{(m)} \rangle + \sum_{r=3}^m \sum_{\mathbf{m}} C'_{r,\mathbf{m}} u^{(m_1)} \dots f^{(m_i)} \dots u^{(m_r)} dx,$$

where  $m_1 + \dots + m_r = 6 + 2m - 2r$ . Using (5.2) with a suitable  $\delta$ , we get

$$I_1 \leq \|u\|_m^2 + C \|u\|_0^{C_1} \leq \|u\|_m^2 + C(1 + \|u\|_0^{4m}) + \|f\|_m^2. \quad (5.3)$$

If  $u(t) = u(t, x)$  is a solution of equation (0.1), then

$$\frac{d}{dt} \mathcal{J}_m(u) = \langle \nabla \mathcal{J}_m(u), \epsilon f \rangle \leq \epsilon \|u\|_m^2 + \epsilon C(1 + \|u\|_0^{4m}) + \epsilon \|f\|_m^2,$$

and

$$\frac{1}{2} \|u\|_m^2 - C(1 + \|u\|_0^{4m}) \leq \mathcal{J}_m(u) \leq 2 \|u\|_m^2 + C(1 + \|u\|_0^{4m}).$$

Denote  $C_m = C(1 + \|u(0)\|_0^{4m}) + C\|f\|_m^2$ , then from (5.1) and above, we deduce

$$\frac{d}{dt} (\mathcal{J}_m(u) - C_m) \leq \frac{1}{2} \epsilon (\mathcal{J}_m(u) - C_m),$$

thus

$$\mathcal{J}_m(u) - C_m \leq e^{\frac{1}{2}\epsilon t} [\mathcal{J}_m(u(0)) - C_m],$$

so

$$\|u(t)\|_m^2 \leq 4 \|u(0)\|_m^2 e^{\frac{1}{2}\epsilon t} + C_m.$$

This prove Proposition 0.3.  $\square$

## Appendix

Consider the following system of ordinary differential equations:

$$\dot{x} = Y(x), \quad x(0) = x_0 \in \mathbb{R}^n,$$

where  $Y(x) = (Y_1(x), \dots, Y_n(x)) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuously differentiable map. Let  $F(t, x)$  be a (local) flow determined by this equation.

**Theorem A** (Liouville). Let  $B(x_1, \dots, x_n)$  be a continuous differentiable function on  $\mathbb{R}^n$ . For the Borel measure  $d\mu = B(x)dx$  in  $\mathbb{R}^n$  and any bounded open set  $A \subset \mathbb{R}^n$ , we have

$$\frac{d}{dt} \mu(F(t, A)) = \int_{F(t, A)} \left[ \sum_{i=1}^n \frac{\partial(B(x)Y_i(x))}{\partial x_i} \right] dx, \quad t \in (-T, T),$$

where  $T > 0$  is such that  $F(t, x)$  is well defined and bounded for any  $t \in (-T, T)$  and  $x \in A$ .

For  $B = \text{const}$  this result is well known. For its proof for a non-constant density  $B$  see e.g. [14, 10].

## Acknowledgement

First of all, I want to thank my PhD supervisor S. Kuksin for formulation of the problem and guidance. I am grateful to A. Boritchev for useful suggestions. Finally, I would like to thank all of the staff and faculty at CMLS of Ecole Polytechnique for help.



## Reference

- [1] Kappeler T and Pöschel J. *KAM & KdV*. Springer, 2003.
- [2] Kuksin S. *Analysis of Hamiltonian PDEs*. Oxford University Press, Oxford, 2000.
- [3] Lochak P and Meunier C. *Multiphase Averaging for Classical systems (with Applications to Adiabatic Theorems)*. Springer, 1988.
- [4] Neistadt A I. Some resonance problem in nonlinear systems. *Thesis, Moscow University*, 1975.
- [5] Neistadt A I. Averaging in multi-frequency systems, I and II. *Soviet Phys, Doklady*, 20(7) and 21(2):492–494 and 80–82, 1975.
- [6] Nekhoroshev N. An exponential estimate of the time of stability of nearly-integrable hamiltonian systems I. *Uspekhi Mat. Nauk*, 32(6):5–66, 1972.
- [7] Kuksin S and Piatnitski A. Khasminskii-whitham averaging for randomly perturbed KdV equation. *J. Math. Pures Appl.*, 89:400–428, 2008.
- [8] Kuksin S. Damped-driven KdV and effective equations for long-time behaviour of its solutions. *Geometric and Functional Analysis*, 20(6):1431–1463, 2010.
- [9] Dudley R M. *Real Analysis and Probability*. Cambridge University Press, 2002.
- [10] Zhidkov P E. *Korteweg-de Vries and Nonlinear Schrödinger Equations: Qualitative Theory*. Springer, 2001.
- [11] Temirgaliev N. A connection between inclusion theorems and the uniform convergence of multiple fourier series. *Matematicheskie Zametki*, 12(2):139–148, 1972.
- [12] Bidegaray B. Invariant measures for some partial differential equations. *Physica D*, 82:340–364, 1995.
- [13] Agrachev A, Kuksin S, Sarychev A, and Shirikyan A. On finite-dimensional projections of distributions for solutions of randomly forced 2D Navier-Stokes equations. *Ann. I. H. Poincare*, PR43:300–315, 2007.
- [14] Kornfeld I P, Formin S V, and Sinai G. *Ergodic Theory*. Springer, 1982.